τ-polynomials of Special Graphs and Their Zeros

Tan Bing Dun
Supervisor: Assoc. Prof. Dong Fengming

National Institute of Education

Abstract

This paper aims to find the τ-polynomial of some special graphs. These include null graphs, trees, cycles, some families of complete bipartite graphs and finally complete graphs. We shall also make some observations with regards to the τ-polynomial of these graphs. With these τ-polynomial, we shall further examine the zeros of them, i.e. the solution of these polynomials when they are equated to zero.

Chapter 1

Introduction

Chromatic polynomial of graphs has been extensively studied in the area of graph theory for years. It enables us to count the number of proper k-colourings of graph G whenever k is a positive integer.

Let $\Pi(G)$ be the set of partitions of $V(G)$. We let $G(P)$ be the spanning subgraph of G with edge set $\{uv \in E(G) : u, v \in P_j \in P\}$, for any $P \in \Pi(G)$. P_j is one set in the partition P. Hence, $G(P)$ is the spanning subgraph obtained from G by removing all edges whose ends are not in the same set of P [Dong [1]].

Lemma 1.1 (Brenti [2]) *For any $i \leq p$,*

$$c_i(G) = \sum_{\substack{P \in \Pi(G) \\ |P|=i}} |A(G(P)|$$

where $A(H)$ is the set of acyclic orientations of a graph H.

This allows us a way of finding the number of acyclic orientations of a graph, if we are able to find the τ-polynomial of the graph, in any number of components.

All these will lead to the discussion of the zeros of τ-polynomials of graphs where we observe that these zeros are all real values up to a certain order. We wish to examine if this is indeed true for all graphs, and for a beginning, at least a certain family of special graphs such as null graphs.

Chapter 2

τ-polynomial of Special Graphs

In this chapter, we shall derive the τ-polynomial of some special graphs that will be used to determine the τ-polynomial of these graphs up to order 8 in the next chapter. We will build these polynomial from the definitions and theorems stated and proven in the preliminaries section.

We will also be making some observations about the τ-polynomials of these special graphs and try to give a proof for the observations.

2.1 Preliminaries

For any graph G, $\tau(G, x)$ is defined as

$$\tau(G, x) = \sum_{0 \leq i \leq p} c_i x^i.$$

The c_i's are determined by the chromatic polynomial

$$\chi(G, x) = \sum_{0 \leq i \leq p} (-1)^{p-i} c_i \langle x \rangle_i,$$

where $\langle x \rangle_i = x(x+1)...(x+i-1)$.

Observation 2.1 *From the definition above, we make the following general observations applicable to all graphs so we shall not need to re-state for special families of graphs which will be discussed later.*

(i). *Coefficients of $\tau(G, x)$ is always positive.*

(ii). *The constant terms of $\tau(G, x)$ is always zero.*

(iii). *The number of terms in the τ-polynomial of a graph of order n is always n.*

(iv). *The leading term of the τ-polynomial of a graph of order n is always x^n.*

(v). *The coefficient of the leading term of the τ-polynomial of a graph of order n is always 1.*

Theorem 2.1 (Dong [1]) *Let N_p be any null graph of order p. The chromatic polynomial is given by*

$$\chi(N_p, x) = x^p = \sum_{1 \le i \le p} (-1)^{p-i} S(p, i) \langle x \rangle_i.$$

Proof. By substituting x with $-z$ in the identity

$$x^p = \sum_{i \le p} S(p, i)(x)_i,$$

we get

$$(-z)^p = \sum_{i \le p} S(p, i)(-z)_i = \sum_{i \le p} (-1)^i S(p, i) \langle z \rangle_i.$$

Therefore

$$\chi(N_p, x) = x^p = \sum_{1 \le i \le p} (-1)^{p-i} S(p, i) \langle x \rangle_i.$$

□

From this, we obtain the τ-polynomial of null graphs, i.e.

$$\tau(N_p, x) = B_p(x),$$

where Bell Polynomial B_p is of the form $B_p(x) = \sum_{i=1}^{p} S(p, i) x^i$.

Example 2.1 *Some examples of τ-polynomial of null graphs N_p.*

(i). $\tau(N_1, x) = x$

(ii). $\tau(N_2, x) = x^2 + x$

(iii). $\tau(N_3, x) = x^3 + 3x^2 + x$

(iv). $\tau(N_4, x) = x^4 + 6x^3 + 7x^2 + x$

(v). $\tau(N_5, x) = x^5 + 10x^4 + 25x^3 + 15x^2 + x$

(vi). $\tau(N_6, x) = x^6 + 15x^5 + 65x^4 + 90x^3 + 31x^2 + x$

(vii). $\tau(N_7, x) = x^7 + 21x^6 + 140x^5 + 350x^4 + 301x^3 + 63x^2 + x$

(viii). $\tau(N_8, x) = x^8 + 28x^7 + 266x^6 + 1050x^5 + 1701x^4 + 966x^3 + 127x^2 + x$

Observation 2.2 *From the example above, we make the following observations.*

(i). *The coefficient of x is always 1.*
Proof.
From the combinatorial identities of Stirling Numbers of the Second Kind, we know that $S(p, 1) = 1$. □

(ii). *The coefficient of x^{n-1} for $\tau(N_p, x)$ for $p \geq 2$ is $\frac{p(p-1)}{2}$.*
Proof.
From the combinatorial identities of Stirling Numbers of the Second Kind, we know that $S(p, p-1) = \binom{p}{2}$. □

(iii). *The difference between coefficient of x^2 for p^{th} and twice the coefficient of the $(p-1)^{th}$ term is always 1.*
Proof.
From the combinatorial identities of Stirling Numbers of the Second Kind, we know that $S(r, n) = S(r-1, n-1) + nS(r-1, n)$. Now, by letting $r = p$ and $n = 2$ we have

$$S(p, 2) = S(p-1, 1) + 2S(p-1, 2).$$

By rearranging the equation and noting that $S(p-1, 1) = 1$, we have

$$S(p, 2) - 2S(p-1, 2) = 1.$$

□

2.2 Trees

In this section, we shall examine the τ-polynomial of trees. Since the chromatic polynomial of trees is equal to that of the chromatic polynomial of paths, we would be able to find the τ-polynomial of paths using the τ-polynomial of trees.

Theorem 2.2 *Let T_n be any tree of order n. Then*

$$\tau(T_n, x) = \sum_{k=1}^{n} \binom{n-1}{k-1} B_k(x).$$

Proof.

$$\chi(T_n, x) = x(x-1)^n - 1 = x\sum_{i=0}^{n-1}(-1)^{n-1-i}\binom{n-1}{i}x^i.$$

Let $k = i + 1$, i.e. $i = k - 1$.

$$\chi(T_n, x) = x\sum_{i=0}^{n}\binom{n-1}{k-1}x^{k-1}(-1)^{n-1-i} = \sum_{i=0}^{n}\binom{n-1}{i-1}x^i(-1)^{n-1-i}$$

This implies

$$\tau(T_n, x) = \sum_{k=1}^{n}\binom{n-1}{k-1}B_k(x)$$

□

Example 2.2 *Some examples of τ-polynomial of trees T_n.*

(i). $\tau(T_1, x) = x$

(ii). $\tau(T_2, x) = x^2 + 2x$

(iii). $\tau(T_3, x) = x^3 + 5x^2 + 4x$

(iv). $\tau(T_4, x) = x^4 + 9x^3 + 19x^2 + 8x$

(v). $\tau(T_5, x) = x^5 + 14x^4 + 55x^3 + 65x^2 + 16x$

(vi). $\tau(T_6, x) = x^6 + 20x^5 + 125x^4 + 285x^3 + 211x^2 + 32x$

(vii). $\tau(T_7, x) = x^7 + 27x^6 + 245x^5 + 910x^4 + 1351x^3 + 665x^2 + 64x$

(viii). $\tau(T_8, x) = x^8 + 35x^7 + 434x^6 + 2380x^5 + 5901x^4 + 6069x^3 + 2059x^2 + 128x$

Observation 2.3 *From the example above, we make the following observations.*

(i). *The coefficient of x is 2^{n-1} for $\tau(T_n, x)$.*
Proof.
Since a tree does not contain any cycles, we only need to consider the number of directions for each edge. $v(T_n) = n - 1$ implies the coefficient of x is 2^{n-1} □

(ii). *The coefficient of x^{n-1} for $\tau(T_n, x)$ forms a quadratic sequence for $n \geq 2$ with first term 2 and the difference between consecutive terms forms an arithmetic progression of first term 3 and common difference 1.*

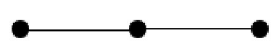
We shall now introduce the Deletion-Contraction formula for τ-polynomials and apply this on trees, with consideration of the previously known results for null graph to verify what we have found in the previous example.

Proposition 2.1 *For any graph G with $e \in E(G)$,*

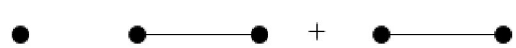
$$\tau(G, x) = \tau(G \setminus e, x) + \tau(G/e, x).$$

For a proof of this, one may wish to refer to the proof written by Dong (2017) in his paper. An illustration of the proposition is as follows.

Consider a tree of order 3.



becomes



becomes

This means T_3 can be reduced into $N_3 + 2N_2 + N(1)$ and the Bell Polynomial of that would be

$$\begin{aligned} &\tau(N_3 + 2N_2 + N(1), x) \\ &= B_3(x) + B_2(x) + B_1(x) \\ &= x^3 + 3x^2 + x + 2x^2 + 2x + x \\ &= x^3 + 5x^2 + 4x \\ &= \tau(T_3, x) \end{aligned}$$

Hence, we have verified the result, using the Deletion-Contraction formula, for τ-polynomials for tree of order 3.

2.3 Cycles

We shall begin this section by introducing a proposition which we will be using to convert the chromatic polynomial of a graph to its τ-polynomial.

Proposition 2.2 (Dong [1]) *For any simple graph G of order p, if*

$$\chi(G,x) = \sum_{k\leq p}(-1)^{p-k}b_k x^k,$$

then

$$\tau(G,x) = \sum_{k\leq p} b_k B_k(x).$$

We shall now proceed to prove the result for the τ-polynomial of a cycle.

Theorem 2.3 *Let C_n be any cycle of order n. Then*

$$\tau(C_n,x) = \sum_{k=1}^{n}\binom{n}{k}B_k(x) - B_1(x).$$

Proof.

$$\chi(C_n,x) = (x-1)^n + (-1)^n(x-1)$$

$$= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}x^i + (-1)^n x + (-1)^{n+1}$$

$$= x^n - \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} - ... + (-1)^{n-2}\binom{n}{2}x^2 + (-1)^{n-1}\binom{n}{1}x + (-1)^n x$$

$$= x^n - \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} - ... + (-1)^{n-2}\binom{n}{2}x^2 + (-1)^{n-1}\binom{n}{1}x - x$$

Hence, by Proposition 2.2, we have

$$\tau(C_n,x) = B_n(x) + \binom{n}{n-1}B_{n-1}(x) + ... + \binom{n}{2}B_2(x) + \binom{n}{1}B_1(x) - B_1(x).$$

Summing the terms from 1 to n, we have

$$\tau(C_n,x) = \sum_{k=1}^{n}\binom{n}{k}B_k(x) - B_1(x).$$

□

Example 2.3 *Some examples of τ-polynomial of cycles C_n.*

(i). $\tau(C_2, x) = x^2 + 2x$

(ii). $\tau(C_3, x) = x^3 + 6x^2 + 6x$

(iii). $\tau(C_4, x) = x^4 + 10x^3 + 25x^2 + 14x$

(iv). $\tau(C_5, x) = x^5 + 15x^4 + 65x^3 + 90x^2 + 30x$

(v). $\tau(C_6, x) = x^6 + 21x^5 + 140x^4 + 350x^3 + 301x^2 + 62x$

(vi). $\tau(C_7, x) = x^7 + 28x^6 + 266x^5 + 1050x^4 + 1701x^3 + 966x^2 + 126x$

(vii). $\tau(C_8, x) = x^8 + 36x^7 + 462x^6 + 2646x^5 + 6951x^4 + 7770x^3 + 3025x^2 + 254x$

2.4 $K(2,n)$ Complete Bipartite Graphs

Before we consider the generalized complete bipartite graphs, we shall first examine complete bipartite graphs with two and three vertices in one set. In particular, we shall examine the family of $K(2,n)$ complete bipartite graphs in this section. We shall also give a proof for the chromatic polynomial of $K(2,n)$ graphs.

Theorem 2.4 *Let $K(2,n)$ be any complete bipartite graph of order $n+2$. The chromatic polynomial of the graph is given by*

$$\chi(K(2,n),x) = x(x-1)(x-2)^n + x(x-1)^n.$$

Proof.

Let the two vertices be u_1 and u_2. We shall consider two cases.

Case 1: u_1 and u_2 are coloured with the same colour.

Clearly, the chromatic polynomial is $x(x-1)^n$.

Case 2: u_1 and u_2 are coloured with different colours.

Clearly, the chromatic polynomial is $x(x-1)(x-2)^n$.

Therefore, by combining the two cases we have

$$\chi(K(2,n),x) = x(x-1)(x-2)^n + x(x-1)^n.$$

□

Next, we shall proceed to find the τ-polynomial of $K(2,n)$ complete bipartite graphs using the chromatic polynomial we have found earlier.

Let $K(2,n)$ be any complete bipartite graph of order $n+2$. Then

$$\tau(K(2,n),x) = B_{n+2}(x) + \sum_{k=-1}^{n+1} [2^{k+2}\binom{n}{n-k-2} + (2^{k+1}-1)\binom{n}{n-k-1}]B_{n-k}(x).$$

Proof.

$$\chi(K(2,n),x) = x(x-1)(x-2)^n + x(x-1)^n = (x^2-x)\sum_{i=0}^{n}\binom{n}{i}x^i(-2)^{n-i} + x\sum_{i=0}^{n}\binom{n}{i}x^i(-1)^{n-i}$$

$$= \sum_{i=0}^{n}\binom{n}{i}x^{i+2}(-2)^{n-i} - \sum_{i=0}^{n}\binom{n}{i}x^{i+1}(-2)^{n-i} + \sum_{i=0}^{n}\binom{n}{i}x^{i+1}(-1)^{n-i}$$

Next, we consider the coefficients of $x^{n+2}, x^{n+1}, ..., x^{n-k}$ terms.

Coefficient of $x^{n+2} = \binom{n}{n} = 1$.

Coefficient of $x^{n+1} = -[2\binom{n}{n-1} + \binom{n}{n} - \binom{n}{n}]$

Coefficient of $x^{n} = 2^2\binom{n}{n-2} + 2\binom{n}{n-1} - \binom{n}{n-1}$

In general, the coefficient of $x^{n-k} = 2^{k+2}\binom{n}{n-k-2} + (2^{k+1}-1)\binom{n}{n-k-1}$

Therefore by Proposition 2.2, we have

$$\tau(K(2,n),x) = B_{n+2}(x) + \sum_{k=-1}^{n+1}[2^{k+2}\binom{n}{n-k-2} + (2^{k+1}-1)\binom{n}{n-k-1}]B_{n-k}(x).$$

□

Example 2.4 *Some examples of τ-polynomial of $K(2,n)$ complete bipartite graphs.*

(i). $\tau(K(2,1),x) = x^3 + 5x^2 + 4x$

(ii). $\tau(K(2,2),x) = x^4 + 10x^3 + 25x^2 + 14x$

(iii). $\tau(K(2,3),x) = x^5 + 16x^4 + 76x^3 + 119x^2 + 46x$

(iv). $\tau(K(2,4),x) = x^6 + 23x^5 + 173x^4 + 508x^3 + 541x^2 + 146x$

(v). $\tau(K(2,5),x) = x^7 + 31x^6 + 335x^5 + 1560x^4 + 3136x^3 + 2375x^2 + 454x$

(vi). $\tau(K(2,6),x) = x^8 + 40x^7 + 584x^6 + 3925x^5 + 12621x^4 + 18340x^3 + 10165x^2 + 1394x$

From the example above, we can compare the τ-polynomials to other special graphs. For example, $K(2,1)$ is isomorphic to a tree of order 3 and $K(2,2)$ is isomorphic to the cycle C_4. By comparing the earlier results, we observe that the τ-polynomial of $K(2,1)$ and $K(2,2)$ are equal to the τ-polynomial of a tree of order 3 and that of a cycle C_4 respectively.

2.5 $K(3,n)$ Complete Bipartite Graphs

We shall now consider $K(3,n)$ complete bipartite graph. The proof for the chromatic polynomial of a $K(3,n)$ graph is similar to that of the $K(2,n)$ graph by considering three cases, i.e. consider the cases where the three vertices are coloured with the same colour, different colours or two of the three vertices coloured with the same colour. Thus, we shall not elaborate on a proof for the chromatic polynomial for such graphs. We shall consider the τ-polynomial for a $K(3,n)$ graph and give a proof for it.

Theorem 2.5 *Let $K(3,n)$ be any complete bipartite graph of order $n+3$. Then*

$$\tau(K(3,n),x) = B_{n+3}(x)$$

$$+\sum_{k=-2}^{n+1}[3^{k+3}\binom{n}{n-k-3} + (3^{k+3}-3(2^{k+2}))\binom{n}{n-k-2} + (2(3^{k+1})-3(2^{k+1})+1]B_{n-k}(x).$$

Proof.

$$\chi(K(3,n),x) = (x^3-3x^2+2x)\sum_{i=0}^{n}\binom{n}{i}x^i(-3)^{n-i}$$

$$+(3x^2-3x)\sum_{i=0}^{n}\binom{n}{i}x^i(-2)^{n-i} + x\sum_{i=0}^{n}\binom{n}{i}x^i(-1)^{n-i}$$

$$=\sum_{i=0}^{n}x^{i+3}(-3)^{n-i} - 3\sum_{i=0}^{n}x^{i+2}(-3)^{n-i} + 2\sum_{i=0}^{n}x^{i+1}(-3)^{n-i}$$

$$+3\sum_{i=0}^{n}x^{i+2}(-2)^{n-i} - 3\sum_{i=0}^{n}x^{i+3}(-2)^{n-i} + \sum_{i=0}^{n}x^{i+1}(-1)^{n-i}$$

Next, we consider the coefficients of $x^{n+3}, x^{n+2}, ..., x^{n-k}$ terms.

Coefficient of $x^{n+3} = \binom{n}{n} = 1$.

Coefficient of $x^{n+2} = (-3)^1\binom{n}{n-1} - 3\binom{n}{n} + 3\binom{n}{n}$

Coefficient of $x^{n+1} = (-3)^2\binom{n}{n-2} - 3\binom{n}{n-1} + 3(-2)^1\binom{n}{n-1} + 2(-3)^0\binom{n}{n} - 3(-2)^0\binom{n}{n} + (-1)^0\binom{n}{n}$

From the above, we find the coefficient of the general term x^{n-k} to be

$$(-3)^{k+3}\binom{n}{n-k-3} + (-3)^{k+3}\binom{n}{n-k-2} + 3(-2)^{k+2}\binom{n}{n-k-2}$$

$$+2(-3)^{k+1}\binom{n}{n-k-1} - 3(-2)^{k+1}\binom{n}{n-k-1} + (-1)^{k+1}\binom{n}{n-k-1}$$

Note: From this we observe that for the six terms in the coefficient, the powers observe the pattern of even, even, odd, even, even, even or odd, odd, even, odd, odd, odd which helps us to decide the polarity of the terms in the subsequent part where we find the expression for the τ-polynomial of $K(3,n)$.

Therefore by Proposition 2.2, we have

$$\tau(K(3,n),x) = B_{n+3}(x)$$

$$+\sum_{k=-2}^{n+1}[3^{k+3}\binom{n}{n-k-3}+(3^{k+3}-3(2^{k+2}))\binom{n}{n-k-2}+(2(3^{k+1})-3(2^{k+1})+1)\binom{n}{n-k-1}]B_{n-k}(x).$$

□

Example 2.5 *Some examples of τ-polynomial of $K(3,n)$ complete bipartite graphs.*

(i). $\tau(K(3,1),x) = x^4 + 9x^3 + 19x^2 + 8x$

(ii). $\tau(K(3,2),x) = x^5 + 16x^4 + 76x^3 + 119x^2 + 46x$

(iii). $\tau(K(3,3),x) = x^6 + 24x^5 + 191x^4 + 606x^3 + 721x^2 + 230x$

(iv). $\tau(K(3,4),x) = x^7 + 33x^6 + 386x^5 + 1992x^4 + 4594x^3 + 4211x^2 + 1066x$

(v). $\tau(K(3,5),x) = x^8 + 43x^7 + 686x^6 + 5150x^5 + 19071x^4 + 33354x^3 + 23809x^2 + 4718x$

From the example above, we can compare the τ-polynomials to other special graphs. For example, $K(3,1)$ is isomorphic to a tree of order 4 and $K(3,2)$ is isomorphic to $K(2,3)$. By comparing the earlier results, we observe that the τ-polynomial of $K(3,1)$ and $K(3,2)$ are equal to the τ-polynomial of a tree of order 4 and $K(2,3)$ respectively.

2.6 $K(m,n)$ Complete Bipartite Graphs

The chromatic polynomial of the general complete bipartite graph is given below

$$\chi(K(m,n),x) = \sum_{r=1}^{m}\sum_{s=1}^{n} S(m,r)S(n,s)(x)_{r+s} = \sum_{r+s=k} S(m,r)S(n,s)(x)_k.$$

In general,

$$(x)_{r+s} = x(x-1)(x-2)...(x-r-s+1) = \sum_{i=1}^{r+s} \begin{bmatrix} r+s \\ i \end{bmatrix} x^i.$$

We can covert this to an expression in terms of the Bell polynomial, i.e.

$$(x)_{r+s} = \sum_{k=1}^{r+s} \begin{bmatrix} r+s \\ k \end{bmatrix} B_k(x).$$

Hence, we have

$$\tau(K(m,n),x) = \sum_{i=1}^{m}\sum_{j=1}^{n} (-1)^{m+n-i-j} S(m,i)S(n,j) \sum_{k=1}^{i+j} (-1)^{i+j-k} \begin{bmatrix} i+j \\ k \end{bmatrix} B_k(x).$$

We shall apply the result, without proof, to verify the results for the τ-polynomial of $K(2,n)$ and $K(3,n)$ to be true.

$BP := (p, q) \to sum\left(sum\left((-1)^{p+q-s-t}\,\mathrm{Stirling2}(q, t)\cdot \mathrm{Stirling2}(p, s)\cdot M(s+t), t = 1 .. q\right), s = 1 .. p\right);$

$$BP := (p, q) \mapsto \sum_{s=1}^{p}\sum_{t=1}^{q} (-1)^{p+q-s-t}\,\mathrm{Stirling2}(q, t)\,\mathrm{Stirling2}(p, s)\,M(s+t)$$

for q **from** 1 **to** 6 **do** $q, BP(2, q)$ **od**;

$$1,\ x^3 + 5x^2 + 4x$$
$$2,\ x^4 + 10x^3 + 25x^2 + 14x$$
$$3,\ x^5 + 16x^4 + 76x^3 + 119x^2 + 46x$$
$$4,\ x^6 + 23x^5 + 173x^4 + 508x^3 + 541x^2 + 146x$$
$$5,\ x^7 + 31x^6 + 335x^5 + 1560x^4 + 3136x^3 + 2375x^2 + 454x$$
$$6,\ x^8 + 40x^7 + 584x^6 + 3925x^5 + 12621x^4 + 18340x^3 + 10165x^2 + 1394x$$

for q **from** 1 **to** 5 **do** $q, BP(3, q)$ **od**;

$$1,\ x^4 + 9x^3 + 19x^2 + 8x$$
$$2,\ x^5 + 16x^4 + 76x^3 + 119x^2 + 46x$$
$$3,\ x^6 + 24x^5 + 191x^4 + 606x^3 + 721x^2 + 230x$$
$$4,\ x^7 + 33x^6 + 386x^5 + 1992x^4 + 4594x^3 + 4211x^2 + 1066x$$
$$5,\ x^8 + 43x^7 + 686x^6 + 5150x^5 + 19071x^4 + 33354x^3 + 23809x^2 + 4718x$$

Example 2.6 *Some examples of τ-polynomial of $K(4,n)$ complete bipartite graphs.*

(i). $\tau(K(4,2),x) = x^6 + 23x^5 + 173x^4 + 508x^3 + 541x^2 + 146x$

(ii). $\tau(K(4,3),x) = x^7 + 33x^6 + 386x^5 + 1992x^4 + 4594x^3 + 4211x^2 + 1066x$

(iii). $\tau(K(4,4),x) = x^8 + 44x^7 + 722x^6 + 5614x^5 + 21741x^4 + 40318x^3 + 31153x^2 + 6902x$

Example 2.7 *Some examples of τ-polynomial of $K(5,n)$ complete bipartite graphs.*

(i). $\tau(K(5,2),x) = x^7 + 31x^6 + 335x^5 + 1560x^4 + 3136x^3 + 2375x^2 + 454x$

(ii). $\tau(K(5,3),x) = x^8 + 43x^7 + 686x^6 + 5150x^5 + 19071x^4 + 33354x^3 + 23809x^2 + 4718x$

2.7 Complete Graphs

We shall use the following proposition to find the τ-polynomial of complete graphs using induction.

Proposition 2.3 (Dong [1]) *Let n be a simplicial vertex of G with degree k, then*

$$\tau(G, x) = x\tau\text{'}(G - u, x) + (x + k)\tau(G - u, x).$$

We shall formally state the result for the τ-polynomial of a complete graph.

Theorem 2.6 *Let K_n be any complete graph of order n. Then*

$$\tau(K_n, x) = \sum_{k \leq n} \begin{bmatrix} n \\ k \end{bmatrix} B_k(x) = \sum_{i \leq n} \sum_{i \leq k \leq n} S(k, i) \begin{bmatrix} n \\ k \end{bmatrix} x^i.$$

Proof. Since $n = 1$ is a null graph, it is trivial.

Suppose the result holds for $p \leq N - 1$, i.e.

$$\tau(K_n, x) = \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} S(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^i.$$

Next, we shall apply Proposition 2.1 to prove the case for $p = N$. Since graph K_N has vertices of degree $N - 1$, we have $k = N - 1$.

Thus, we have

$$\tau\text{'}(K_{N-1}, x) = \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} iS(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^{i-1}.$$

Multiplying both sides by x, we have

$$x\tau\text{'}(K_{N-1}, x) = \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} iS(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^i.$$

Next, we also have

$$(x + N - 1)\tau(K_{N-1}, x) = (x + N - 1) \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} S(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^i.$$

By combining the above two, based on Proposition 2.1, we obtain

$$\tau(K_N, x) = \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} iS(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^i + \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} (N-1)S(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^i + \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} S(k, i) \begin{bmatrix} N-1 \\ k \end{bmatrix} x^{i+1}$$

We shall compare, term by term, the above to what we wish to prove, i.e.

$$\sum_{i=1}^{N}\sum_{k=i}^{N} S(k,i)\begin{bmatrix}N\\k\end{bmatrix}x^i.$$

First, we consider the coefficient of the x^N term.

$$S(N-1,N-1)\begin{bmatrix}N-1\\N-1\end{bmatrix} = 1 = S(N,N)\begin{bmatrix}N\\N\end{bmatrix}$$

The above shows that the coefficients of x^N corresponds for both forms.

Next, we shall consider the coefficient of x^j, where $1 \leq j \leq N-1$. We expect to obtain

$$\sum_{k=j}^{N} S(k,j)\begin{bmatrix}N\\k\end{bmatrix}.$$

Next, we have

$$\sum_{k=j}^{N-1} jS(k,j)\begin{bmatrix}N-1\\k\end{bmatrix} + \sum_{k=j-1}^{N-1} S(k,j-1)\begin{bmatrix}N-1\\k\end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix}N-1\\k\end{bmatrix}.$$

By combining the first two terms, we have

$$\sum_{k=j}^{N-1}[S(k+1,j)\begin{bmatrix}N-1\\k\end{bmatrix}] + S(j-1,j-1)\begin{bmatrix}N-1\\j-1\end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix}N-1\\k\end{bmatrix}.$$

This simplifies into

$$\sum_{k=j}^{N-1}[S(k+1,j)\begin{bmatrix}N-1\\k\end{bmatrix}] + \begin{bmatrix}N-1\\j-1\end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix}N-1\\k\end{bmatrix}.$$

Finally, we obtain

$$\sum_{k=j-1}^{N-1} S(k+1,j)\begin{bmatrix}N-1\\k\end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix}N-1\\k\end{bmatrix}.$$

Let $s = k+1$ for the first term, i.e. $k = s-1$, we have

$$\sum_{s=j}^{N} S(s,j)\begin{bmatrix}N-1\\s-1\end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix}N-1\\k\end{bmatrix}.$$

Interchanging s and k, we have

$$\sum_{k=j}^{N} S(k,j)\begin{bmatrix} N-1 \\ k-1 \end{bmatrix} + \sum_{k=j}^{N-1}(N-1)S(k,j)\begin{bmatrix} N-1 \\ k \end{bmatrix}.$$

Since $\begin{bmatrix} N-1 \\ N \end{bmatrix} = 0$, we can rewrite the above as

$$\sum_{k=j}^{N} S(k,j)\begin{bmatrix} N-1 \\ k-1 \end{bmatrix} + \sum_{k=j}^{N}(N-1)S(k,j)\begin{bmatrix} N-1 \\ k \end{bmatrix}$$

$$= \sum_{k=j}^{N} S(k,j)[(N-1)\begin{bmatrix} N-1 \\ k \end{bmatrix} + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}]$$

$$= \sum_{k=j}^{N} S(k,j)\begin{bmatrix} N \\ k \end{bmatrix}.$$

By induction, this proves that the τ-polynomial of complete graphs is

$$\tau(K_n, x) = \sum_{k \leq n} \begin{bmatrix} n \\ k \end{bmatrix} B_k(x) = \sum_{i \leq n} \sum_{i \leq k \leq n} S(k,i) \begin{bmatrix} n \\ k \end{bmatrix} x^i.$$

□

Example 2.8 *Some examples of τ-polynomial of complete graphs K_n.*

(i). $\tau(K_2, x) = x^2 + 2x$

(ii). $\tau(K_3, x) = x^3 + 6x^2 + 6x$

(iii). $\tau(K_4, x) = x^4 + 12x^3 + 36x^2 + 24x$

(iv). $\tau(K_5, x) = x^5 + 20x^4 + 120x^3 + 240x^2 + 120x$

(v). $\tau(K_6, x) = x^6 + 30x^5 + 300x^4 + 1200x^3 + 1800x^2 + 720x$

(vi). $\tau(K_7, x) = x^7 + 42x^6 + 630x^5 + 4200x^4 + 12600x^3 + 15120x^2 + 5040x$

(vii). $\tau(K_8, x) = x^8 + 56x^7 + 1176x^6 + 11760x^5 + 58800x^4 + 141120x^3 + 141120x^2 + 40320x$

From the example above, we can compare the τ-polynomial of K_3 to C_3 as they are isomorphic. We find that their τ-polynomial are indeed equal.

Chapter 3

Zeros of τ-polynomials

From chapter 2.1, we know that the τ-polynomial of trees is

$$\tau(T,x) = \sum_{n=1}^{n} \binom{n-1}{k-1} B_k(x).$$

Conjecture 3.1 (Brenti [2]) *Let G be any graph of order n and $\tau(G,x)$ be the τ-polynomial of graph G. All the roots to the equation $\tau(G,x) = 0$ are real.*

The above conjecture has been verified true for all connected graphs with order of at most 8, in a paper written by F. Brenti (1992). Indeed, in the computation of the roots of the τ-polynomials of special graphs considered in this paper, the roots of the polynomials are all real up to order 10 or more. For example, it is observed that the zeros of the τ-polynomials of null graphs are all real up to order 15, as shown below.

```
> B := n→sum(Stirling2(n, i)·x^i, i = 1..n)
```

$$B := n \mapsto \sum_{i=1}^{n} \mathrm{Stirling2}(n,i)\, x^i$$

```
> for m from 1 to 15 do fsolve(B(m)) od
0.
−1., 0.
−2.618033989, −0.3819660113, 0.
−4.490863615, −1.343379569, −0.1657568157, 0.
−6.510134099, −2.651692996, −0.7621690715, −0.07600383436, 0.
−8.626187889, −4.180787855, −1.703869167, −0.4532506380, −0.03590445077, 0.
−10.81168380, −5.863022111, −2.890457788, −1.140450386, −0.2771164073, −0.01726950400, 0.
−13.04980249, −7.658280786, −4.256556522, −2.071975715, −0.7824237540, −0.1725575136, −0.008403221457, 0.
−15.32938803, −9.540782698, −5.759647508, −3.192778888, −1.518604931, −0.5458249710, −0.1088520768, −0.004120894575, 0.
−17.64262441, −11.49292568, −7.370871513, −4.462998865, −2.443761215, −1.130068321, −0.3853872493, −0.06933098558, −0.002031760027, 0.
−19.98380218, −13.50212136, −9.069797508, −5.853879254, −3.523226505, −1.896751951, −0.8503315329, −0.2745959895, −0.04448817149, −0.001005542393, 0.
−22.34861127, −15.55903240, −10.84143167, −7.344239160, −4.730184039, −2.817538176, −1.487406222, −0.6452926917, −0.1970499476, −0.02871542139,
    −0.0004990082434, 0.
−24.73371012, −17.65652621, −12.67443538, −8.918094305, −6.043999910, −3.868595124, −2.274898873, −1.175677403, −0.4929825562, −0.1422090127,
    −0.01862297506, −0.0002481241616, 0.
−27.13645049, −19.78902109, −14.56001651, −10.56310327, −7.448626367, −5.030657330, −3.192503284, −1.850453190, −0.9351471636, −0.3786648788,
    −0.1031079938, −0.01212488181, −0.0001235532107, 0.
−29.55469440, −21.95205973, −16.49121188, −12.26954124, −8.931395758, −6.288238257, −4.222829582, −2.653046251, −1.514153929, −0.7476465893,
    −0.2921549337, −0.07504603652, −0.007919831359, −0.00006158817753, 0.
```

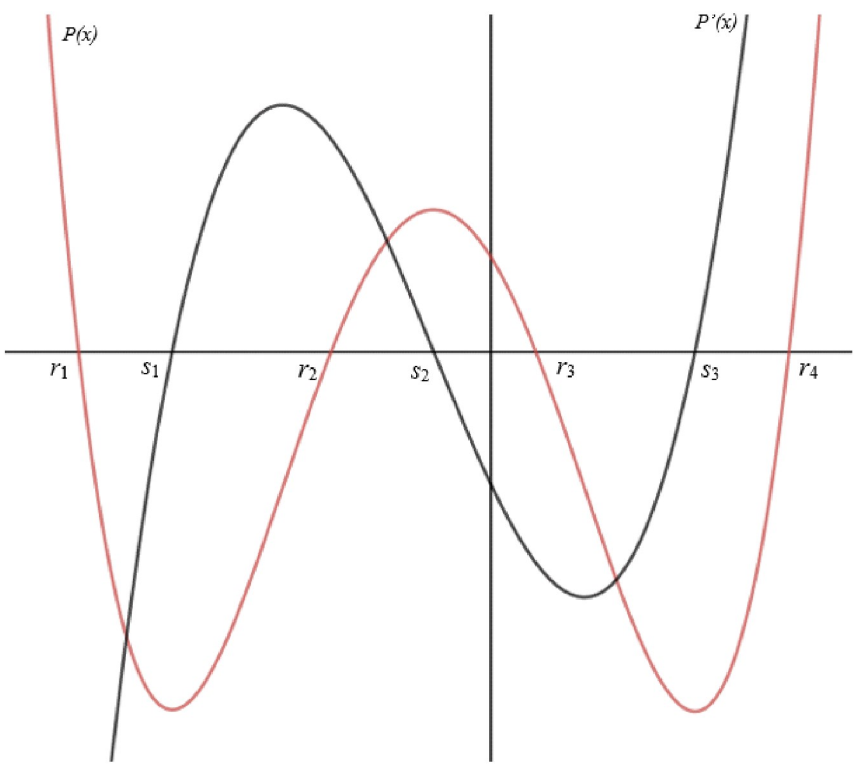
To prove that the zeros of τ-polynomial of null graphs are all real, we need to first prove a result regarding the number of real roots of the sum of a function $P(x)$ and its derivative $P^{'}(x)$.

Theorem 3.1 *Let $P(x)$ be a polynomial of order n in the form $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$. If $P(x)$ has n different real roots, then $P(x) + P^{'}(x)$ also has n different real roots.*

Proof.
We shall first verify that the result holds for order 4, and observe an inductive pattern from there.

Let roots of $P(x)$ be r_i and $P^{'}(x)$ be s_i.

It is clear that for $P(x)$ and $P^{'}(x)$, a root exists in the intervals where either $P(x)$ or $P^{'}(x)$ is positive and the other is negative.



From the diagram above, we can see that a root for $P(x) + P^{'}(x)$ exists in each of the following intervals

$$(s_1, r_2), (s_2, r_3), (s_3, r_4).$$

Since the leading term in $P(x)$ is one order higher than that of $P^{'}(x)$, a root also exists in the interval $(-\infty, r_1)$ for $P(x)+P^{'}(x)$. This implies that there are 4 different roots for $P(x)+P^{'}(x)=0$ when $P(x)$ is of order 4.

Consider $P(x)$ of order n.

By similar reasoning, there exists a real root in the interval $(-\infty, r_1)$. Next, there exists a real root in each of the intervals

$$(s_1, r_2), (s_2, r_3), ..., (s_{n-1}, r_n).$$

Hence, there exist n different roots for $P(x)+P^{'}(x)=0$. □

Next, we shall begin to prove that the zeros of τ-polynomial of null graphs are all real.

Theorem 3.2 *Let N_p be any null graph of order p. The roots of $\tau(N_p, x)=0$ are all real.*

Proof. We shall prove this by induction.

Base case is trivial since the solution to $B_1(x)=0$ is 0.

Next, suppose result holds for $p \leq N-1$.

$$\tau(N_p, x)=0$$

By Proposition 2.1, we have

$$x\tau\text{'}(N_p-u, x)+(x+0)\tau(N_p-u, x).$$

This implies either $x=0$ or

$$\tau\text{'}(N_p-u, x)+\tau(N_p-u, x)=0.$$

Now, N_p-u is the same as N_{p-1}. Therefore, we have

$$\tau\text{'}(N_{p-1}, x)+\tau(N_{p-1}, x)=0,$$

Since we have earlier assumed that the roots of null graphs of order $N-1$ or less are all real, by theorem 3.1, we can also say that the roots of $\tau\text{'}(N_{p-1}, x)+\tau(N_{p-1}, x)=0$ are all real.

This implies there are only real roots for $\tau(N_p, x)=0$. □

Next, we shall turn our focus and numerically examine the roots of τ-polynomials. The use of Maple will help us to quickly find these roots and be able to verify that the result is true up to a certain order. For the case of complete bipartite graphs, we shall focus on $K(2, n)$ and $K(3, n)$ graphs.

Using maple we are able to find that the roots of the τ-polynomials of trees up to order 15, indeed has only real roots as shown in the diagram below.

```
> T := m→sum(binomial(m − 1, k − 1)·B(k), k = 1..m)
```

$$T := m \mapsto \sum_{k=1}^{m} \binom{m-1}{k-1} B(k)$$

```
>
> for m from 1 to 15 do fsolve(T(m)) od
```

$0.$

$-2., 0.$

$-4.000000000, -1.000000000, 0.$

$-6.100431986, -2.338879686, -0.5606883283, 0.$

$-8.273750630, -3.898083197, -1.496707373, -0.3314587997, 0.$

$-10.50206428, -5.606564509, -2.689282494, -1., -0.2020887128, 0.$

$-12.77350372, -7.424154904, -4.064994296, -1.925994664, -0.6856280942, -0.1257243171, 0.$

$-15.07980186, -9.325644597, -5.578494854, -3.047275767, -1.411162453, -0.4782903670, -0.07933010563, 0.$

$-17.41495476, -11.29403404, -7.199868201, -4.320764706, -2.331868345, -1.050119348, -0.3378121575, -0.05057843608, 0.$

$-19.77445197, -13.31724573, -8.908299962, -5.716171651, -3.410325748, -1.809866588, -0.7903161938, -0.2408197507, -0.03250241174, 0.$

$-22.15481024, -15.38635139, -10.68868582, -7.211552090, -4.618205451, -2.727143872, -1.419405201, -0.5999149970, -0.1729158620, -0.02101507646, 0.$

$-24.55327773, -17.49454114, -12.52968969, -8.790531357, -5.934023362, -3.776830068, -2.202009743, -1.122106274, -0.4584650709, -0.1248714622,$
$-0.01365410075, 0.$

$-26.96763852, -19.63648781, -14.42256713, -10.44056759, -7.341240616, -4.938863139, -3.116820237, -1.791328915, -0.8927095559, -0.3522741053,$
$-0.09059595911, -0.008906421947, 0.$

$-29.39607908, -21.80793607, -16.36041806, -12.15183578, -8.826899384, -6.197258921, -4.145764983, -2.590288587, -1.465951557, -0.7138690635,$
$-0.2718891959, -0.06598105102, -0.005828270151, 0.$

$-31.83709436, -24.00542653, -18.33769290, -13.91648887, -10.38067274, -7.539182195, -5.273882520, -3.503213204, -2.164862876, -1.205530968,$
$-0.5732905886, -0.2106297191, -0.04820842042, -0.003824105189, 0.$

Next, we also observe that the roots of the τ-polynomials of cycles up to order 15, indeed have only real roots as shown in the diagram below.

```
C := m→-B(1) + sum(binomial(m, k)·B(k), k = 1..m)
```

$$C := m \mapsto -B(1) + \left(\sum_{k=1}^{m} \binom{m}{k} B(k) \right)$$

```
for m from 1 to 15 do fsolve(C(m)) od
```

$-2., 0.$

$-4.732050808, -1.267949192, 0.$

$-6.470895516, -2.739288614, -0.7898158701, 0.$

$-8.629711838, -4.161957670, -1.723770028, -0.4845604641, 0.$

$-10.81145618, -5.865163613, -2.884336084, -1.142282275, -0.2967618463, 0.$

$-13.04981373, -7.658115722, -4.257382991, -2.070392341, -0.7811946609, -0.1831005582, 0.$

$-15.32938759, -9.540792169, -5.759576249, -3.193033301, -1.518287570, -0.5447449023, -0.1141782244, 0.$

$-17.64262442, -11.49292525, -7.370875991, -4.462974967, -2.443828513, -1.130037108, -0.3847792822, -0.07195446303, 0.$

$-19.98380218, -13.50212138, -9.069797288, -5.853880849, -3.523219638, -1.896767476, -0.8503451716, -0.2742985669, -0.04576744899, 0.$

$-22.34861127, -15.55903239, -10.84143168, -7.344239078, -4.730184518, -2.817536409, -1.487409203, -0.6453047393, -0.1969133563, -0.02933735436, 0.$

$-24.73371012, -17.65652621, -12.67443538, -8.918094309, -6.043999884, -3.868595252, -2.274898458, -1.175677770, -0.4929889094, -0.1421482664,$
$-0.01892543386, 0.$

$-27.13645049, -19.78902109, -14.56001651, -10.56310327, -7.448626368, -5.030657323, -3.192503316, -1.850453102, -0.9351471182, -0.3786677369,$
$-0.1030814316, -0.01227224754, 0.$

$-29.55469440, -21.95205973, -16.49121188, -12.26954124, -8.931395758, -6.288238257, -4.222829580, -2.653046259, -1.514153913, -0.7476465309,$
$-0.2921561271, -0.07503452265, -0.007991809231, 0.$

$-31.98668839, -24.14202119, -18.46240664, -14.02960710, -10.48215527, -7.628818909, -5.351372707, -3.568231818, -2.217144408, -1.245001018,$
$-0.6002854563, -0.2262512917, -0.05479326240, -0.005222528382, 0.$

We observe that the roots of the τ-polynomials of complete graphs up to order 15 also have real roots only.

```
K := s→sum((-1)^(s-r)·Stirling1(s, r)·B(r), r = 1..s)
```

$$K := s \mapsto \sum_{r=1}^{s} (-1)^{s-r}\,\mathrm{Stirling1}(s, r)\, B(r)$$

```
for t from 2 to 15 do fsolve(K(t)) od
```

−2., 0.
−4.732050808, −1.267949192, 0.
−7.758770483, −3.305407289, −0.9358222275, 0.
−10.95389431, −5.731178752, −2.571635008, −0.7432919280, 0.
−14.26010307, −8.399066971, −4.610833151, −2.112965959, −0.6170308533, 0.
−17.64596355, −11.23461043, −6.918816567, −3.876641520, −1.796299810, −0.5276681217, 0.
−21.09217691, −14.19416555, −9.420699383, −5.916297249, −3.352050503, −1.563586190, −0.4610242198, 0.
−24.58595524, −17.24973553, −12.07005513, −8.161709688, −5.181943101, −2.956254556, −1.384963185, −0.4093835732, 0.
−28.11834338, −20.38218199, −14.83591452, −10.56732081, −7.221786539, −4.616882515, −2.646033841, −1.243357962, −0.3681784529, 0.
−31.68280097, −23.57778709, −17.69648757, −13.10172358, −9.428354813, −6.487353031, −4.166840988, −2.395869925, −1.128253356, −0.3345286763, 0.
−35.27439070, −26.82634995, −20.63580567, −15.74226029, −11.77100657, −8.527292008, −5.894911172, −3.799047606, −2.189611942, −1.032797399, −0.3065267021, 0.
−38.88928438, −30.12005863, −23.64178375, −18.47199663, −14.22715236, −10.70738869, −7.792813940, −5.405491020, −3.492354070, −2.016492139, −0.9523260414, −0.2828583482, 0.
−42.52444757, −33.45278497, −26.70503410, −21.27791176, −16.77961364, −13.00562240, −9.832808251, −7.180610494, −4.993575607, −3.232418699, −1.869033815, −0.8835503074, −0.2625883982, 0.
−46.17743002, −36.81962426, −29.81810512, −24.14975795, −19.41499283, −15.40498803, −11.99370343, −9.098302704, −6.661341497, −4.641634067, −3.009132459, −1.741872401, −0.8240822200, −0.2450330151, 0.

Likewise, we find that the roots of τ-polynomials of complete bipartite graphs $K(2, n)$ also have only real roots for order up to 12.

```
K := m→B(m + 2) + sum((2^(k+2)·binomial(m, m − k− 2) + (2^(k+1) − 1)·binomial(m, m − k− 1))·B(m − k), k=-1..m + 1)
```

$$K := m \mapsto B(m+2) + \left(\sum_{k=-1}^{m+1} \left(2^{k+2} \binom{m}{m-k-2} + \left(2^{k+1}-1\right) \binom{m}{m-k-1} \right) B(m-k) \right)$$

```
for m from 1 to 12 do fsolve(K(m)) od
```

−4.000000000, −1.000000000, 0.
−6.470895516, −2.739288614, −0.7898158701, 0.
−8.893700143, −4.536196657, −2., −0.5701032004, 0.
−11.31051766, −6.421656777, −3.390622361, −1.475373692, −0.4018295091, 0.
−13.73249598, −8.379975851, −4.920663085, −2.588832021, −1.095276813, −0.2827562476, 0.
−16.16316233, −10.39759474, −6.559359873, −3.860841391, −2., −0.8191061565, −0.1999355161, 0.
−18.60350417, −12.46424568, −8.285246704, −5.258189138, −3.070203847, −1.559194961, −0.6171416485, −0.1422738536, 0.
−21.05353974, −14.57214982, −10.08283219, −6.757535923, −4.273971851, −2.465018484, −1.224911581, −0.4681652920, −0.1018751123, 0.
−23.51290099, −16.71530598, −11.94056109, −8.341748133, −5.588145174, −3.507782520, −1.994256930, −0.9686387162, −0.3573010661, −0.07335940448, 0.
−25.98107474, −18.88899115, −13.84957106, −9.997825565, −6.995317955, −4.665588804, −2.900528720, −1.623590649, −0.7703039089, −0.2741214643, −0.05308599089, 0.
−28.45750850, −21.08941910, −15.80290539, −11.71564578, −8.482035808, −5.921482878, −3.923812101, −2.412965339, −1.328846272, −0.6155426350, −0.2112576162, −0.03857858410, 0.
−30.94165803, −23.31350448, −17.79499507, −13.48715954, −10.03766068, −7.262083265, −5.048110704, −3.319284862, −2.017507267, −1.092532091, −0.4939212618, −0.1634447224, −0.02813802533, 0.

Lastly, we find that the roots of τ-polynomials of complete bipartite graphs $K(3, n)$ also have only real roots for order up to 12.

```
K := m→B(m + 3) + sum((3^(k+3)·binomial(m, m − k − 3) + 3^(k+3)·binomial(m, m − k − 2) − 3·2^(k+2)·binomial(m, m − k − 2) + 2·3^(k+1)·binomial(m, m − k − 1) − 3·2^(k+1)
    ·binomial(m, m − k − 1) + binomial(m, m − k − 1))·B(m − k), k=−2 ..m + 1)
```

$$K := m \mapsto B(m+3) + \left(\sum_{k=-2}^{m+1} \left(3^{k+3} \binom{m}{m-k-3} + 3^{k+3} \binom{m}{m-k-2} - 3\,2^{k+2} \binom{m}{m-k-2} + 2\,3^{k+1} \binom{m}{m-k-1} - 3\,2^{k+1} \binom{m}{m-k-1} + \binom{m}{m-k-1} \right) B(m-k) \right)$$

```
for m from 1 to 12 do fsolve(K(m)) od
```

−6.100431986, −2.338879686, −0.5606883283, 0.

−8.893700143, −4.536196657, −2., −0.5701032004, 0.

−11.55349678, −6.660658312, −3.622472761, −1.669029476, −0.4943426688, 0.

−14.15869126, −8.798409325, −5.326670971, −2.954772716, −1.362384672, −0.3990710561, 0.

−16.73686411, −10.96084409, −7.106762411, −4.368334152, −2.418660098, −1.096032265, −0.3125028761, 0.

−19.30059672, −13.14888807, −8.951656161, −5.885263265, −3.617262501, −1.978620274, −0.8759289915, −0.2417840260, 0.

−21.85649105, −15.36104775, −10.85169705, −7.487666998, −4.931196972, −3.007833051, −1.618822600, −0.6988897783, −0.1863547452, 0.

−24.40832380, −17.59529982, −12.79907636, −9.162055473, −6.341182423, −4.157679560, −2.508233513, −1.326357471, −0.5581739693, −0.1436176170, 0.

−26.95838226, −19.84963330, −14.78752102, −10.89800091, −7.832704330, −5.409168471, −3.520304670, −2.097446606, −1.089225373, −0.4467585302,
−0.1108545247, 0.

−29.50810605, −22.12220782, −16.81195175, −12.68726736, −9.394467646, −6.747787325, −4.636985010, −2.991076537, −1.758975402, −0.8968876017,
−0.3585285562, −0.08575894867, 0.

−32.05842423, −24.41138732, −18.86820866, −14.52322991, −11.01746903, −8.162099305, −5.844185967, −3.990564601, −2.549492615, −1.479354700,
−0.7405593968, −0.2885150302, −0.06650923311, 0.

−34.60994494, −26.71573220, −20.95284441, −16.40047523, −12.69439690, −9.642883718, −7.130639733, −5.082527625, −3.445780545, −2.179515829, −1.247625995,
−0.6131292718, −0.2327958093, −0.05170779428, 0.

From the discussion above, we can see that the roots of τ-polynomials are all real for order 10 or more. This presents a sort of consistency, considering the earlier proof that all the τ-polynomials of null graphs have only real roots, which points towards a more general result as stated in the conjecture. Proof that the τ-polynomials of other special graphs may involve more complex manipulation which is beyond the scope of this study.

Chapter 4

Conclusion

We have seen various ways of finding the τ-polynomials of some special graphs using various techniques and approaches. Examining these polynomials further, we observe that when we solve for the roots of these polynomials, i.e. zeros.

Each of these τ-polynomials represents a combinatorial perspective of the properties of each graph with respect to its acyclic orientations. In some cases, it is clear that the consideration of the number of components of the special graph is not a straightforward exercise. As such, it may seem difficult to observe any noticeable trend or pattern with respect to the coefficients of certain terms in the τ-polynomials of these graphs. Furthermore, we can observe that as the power of these τ-polynomials get higher, the coefficients become rather large, especially for graphs with appears more complex than others.

In the previous chapter, we have proven that for all null graphs, the τ-polynomials have only real roots. Other families of graph may require more complicated steps in the proofs, especially considering the combinatoric requirement of their τ-polynomials. Hence, in future studies, one may wish to prove that the roots of τ-polynomials of various special graphs are all real. Furthermore, it would be of interest to study if the roots of τ-polynomials of all graphs are real.

Bibliography

[1] Dong, F. (2017, May 26). Polynomials Related to Chromatic Polynomials. Retrieved from NIE MME.

[2] F. Brenti, Expansions of chromatic polynomials and log-concavity, Trans. Amer. Math. Soc. 332 (1992) 729-756.